

Projects for Differential Algebra
Due Friday, December 26, 2014, 11:59 pm EST

Choose one question from the list below, prepare your answer in LaTeX or write it very clearly by hand, produce a PDF of your answer, and send it by e-mail to aovchinnikov@qc.cuny.edu by the due date above. Do not forget to provide all details in the proofs. “One can show...”, “a calculation shows...”, “one may assume...” must be avoided.

1. Find the websites of Alexey Ovchinnikov, Michael Singer, Lucia di Vizio, and Michael Wibmer, download the papers from there dated in the last three years, carefully read their introductions, and, based on this, present a list important open problems in differential algebraic geometry and Differential Galois Theory, explain the importance of the problems, and describe the partial cases in which these problems have been solved.
2. Develop an algorithm with input $a, b \in \mathbb{Q}(x)$ and output the differential Galois group of $y'' + ay' + by = 0$ over the field $\mathbb{C}(x)$ if the Galois group of $z'' - rz = 0$, where $r = a^2/4 + a'/2 - b$, is known.

Hint: read §3.4 and the rest if needed in the paper by Carlos Arreche on the ArXiv:

<http://arxiv.org/abs/1208.2226>.

3. Calculate the differential Galois group of $y'' + 2xy = 0$ over $\mathbb{C}(x)$.
Hint: read Kaplansky and/or Exercise 1.36 in the book by Singer-van der Put.
4. Present a proof of the cyclic vector theorem (if K is a non-constant differential field, then every differential module over K has a cyclic vector) based only on the facts proven in the course (see the lecture notes). Explain why this statement is important for computation with scalar and matrix differential equations.
5. Prove that the differential Galois group G of $Y' = AY$ over K is a subgroup of $SL_n(K^d)$ if and only if the differential equation $y' = \text{trace}(A)y$ has a solution in K .
6. Prove that $[y'^2 - y] : y'^\infty \not\subset [y]$ but $[y'^2 - y^3] : y'^\infty \subset [y]$. Based on this, prove or disprove that
 - (a) $\{y'^2 - y\}$ is prime,
 - (b) $\{y'^2 - y^3\}$ is prime.

Hint: among the papers by Evelyne Hubert, find the right one to use. All algorithms and statements from Hubert's paper that are used must be clearly quoted. It is also recommended to watch this video (talk by Michael Singer at the Kolchin Seminar on November 7, 2014):

<https://videostreaming.qc.cuny.edu/videos/video/2250/in/channel/55/>

7. Let $I \subset K[y_1, \dots, y_n]$ be a prime ideal. Prove that $\{I\} \subset K\{y_1, \dots, y_n\}$ is prime.
Hint: it would be a good idea to review an article by Evelyne Hubert dated 2000, which is available for download at her website, and understand what the statements from there give you if all polynomials in the original input system have order zero.
8. Prove that every autoreduced set is finite. Find the definition of a coherent autoreduced set in Kolchin's book and prove that, in characteristic zero, every characteristic set of a differential ideal is a coherent autoreduced set.
9. Produce a comprehensive list of all misprints (including the mathematical ones) in the course lecture notes.